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VORTEX MOTION BEHIND A CIRCULAR CYLINDER

Ludwig Föppl

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16. Abstract

A theoretical discussion and calculation of vortex motion behind a circular cylinder moving through a fluid, in this case water. It is shown that a pair of vortices form behind a moving cylinder and that their centers will move along a predictable curve. This curve represents an equilibrium condition which, however, is subject to perturbation. The stability of the vortex pair is investigated. Movement of the vortex pair away from the cylinder is calculated as an explanation of the resistance of the cylinder. Finally, the principles elaborated are applied to the flow around a flat plate.

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VORTEX MOTION BEHIND A CIRCULAR CYLINDER

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Göttingen

Presented by S. Finsterwalder at the meeting of January 11, 1913

1. The Equilibrium Condition of the Vortex Pair /1*

If a circular cylinder in a container of water starts to move from a still position, soon after motion commences, two vortices spinning in opposite directions form behind the cylinder. The force of the vortices steadily increases, since fluid is continually accruing from the boundary layer between the edge of the cylinder and the potential current. Meanwhile, the pair of vortices moves away from the cylinder, but at a velocity that is small compared to the velocity through the standing water of the cylinder together with the pair of vortices. In the Gottingen Institute for Applied Mechanics, the various stages of development of the vortices after their detachment were photographed after having been made visible by means of dusting with lycopodium powder.[1] The photographs reproduced in this article depict the vortex pair in an advanced state of development. In the experiment, one makes the assumption that the vortex pair which is completely developed is /2
almost stationary with respect to the cylinder.

The observations and experiments cited prompted Herr v. Kármán and myself to ask whether there are positions behind a circular cylinder moving in still water where the vortices would remain stationary with respect to the cylinder. Concurrently, we hoped to deal with the infinitely large

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1. The photographs were supplied by Herr Rubach who, in his forthcoming dissertation, will publish experimental material pertaining to the subject that is herein treated theoretically. Herr Rubach has kindly supplied me with the accompanying photographs.

* Numbers in the margin indicate pagination in the foreign text.

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velocity of the potential current on either end of plates moving through still water that likewise generate pairs of vortices.[2]

I attempted to answer this question by searching for the stationary site of the vortex pair behind a circular cylinder of the radius 1. If the complex coordinates

$$\zeta = \xi + i\eta$$

describe the flow in the plane bound to the cylinder, and the position of the vortex is

$$\zeta_0 = \xi_0 + i\eta_0 \quad \text{or} \quad \bar{\zeta}_0 = \xi_0 - i\eta_0,$$

then the complex velocity potential W can be written with the help of the two vortices in the following manner:

$$W = \Phi + i\Psi = U\left(\zeta + \frac{1}{\zeta}\right) + iC \lg \frac{(\zeta - \zeta_0)\left(\zeta - \frac{1}{\zeta_0}\right)}{(\zeta - \bar{\zeta}_0)\left(\zeta - \frac{1}{\bar{\zeta}_0}\right)}, \tag{1}$$

in which means the velocity in the infinite, and C the force of the vortex. Φ is the velocity potential; $\Psi =$ a constant are the streamlines. By means of differentiation, one obtains for (1) the complex velocity

$$\frac{dW}{d\zeta} = u - iv = U\left(1 - \frac{1}{\zeta^2}\right) + iC\left\{\frac{1}{\zeta - \zeta_0} + \frac{1}{\zeta - \frac{1}{\zeta_0}} - \frac{1}{\zeta - \bar{\zeta}_0} - \frac{1}{\zeta - \frac{1}{\bar{\zeta}_0}}\right\}. \tag{2}$$

In order to derive the velocity of the two vortices which naturally lie in mirror image across the ξ-axis, one substitutes ζ with ζ_0 in (2). The term /3

$$iC\frac{1}{\zeta - \zeta_0}$$

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2. This case has already been dealt with years ago by Prof. W. Kutta. An extensive characterization of the flow can be found in the collection of the Mathematical Institute of the Technischen Hochschule in Munich. (Footnote by Prof. Finsterwalder.)

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gives an infinitely large velocity in the center of the individual vortex for $\zeta = \zeta_0$, which, however, contributes nothing to the displacement velocity of the vortex center and so must be ignored here. The complex displacement velocity of the vortex center is therefore:

$$u_0 - iv_0 = U\left(1 - \frac{1}{\zeta_0^2}\right) + iC\left\{\frac{1}{\zeta_0 - \frac{1}{\zeta_0}} - \frac{1}{\zeta_0 - \bar{\zeta}_0} - \frac{1}{\zeta_0 - \frac{1}{\bar{\zeta}_0}}\right\}. \tag{3}$$

If we now leave out the indices 0 as understood and divide the equation (3) into real and imaginary parts, then we obtain the two equations

$$\begin{aligned} &\text{(a)} \quad u = U\left(1 - \frac{\xi^2 - \eta^2}{r^4}\right) + C\eta\left(\frac{1}{(r^2 - 1)^2} - \frac{1}{2\eta^2}\right), \\ &\text{(b)} \quad -v = 2U\frac{\xi\eta}{r^4} - C\frac{\xi}{r^4 - 1}, \end{aligned} \tag{4}$$

where $r^2 = \xi^2 + \eta^2$. Therefore, the condition for the pair of vortices to remain stationary is simply $u = 0$ and $v = 0$, or:

$$\begin{aligned} &\text{(a)} \quad U\left(1 - \frac{\xi^2 - \eta^2}{r^4}\right) + C\eta\left(\frac{1}{(r^2 - 1)^2} - \frac{1}{2\eta^2}\right) = 0 \\ &\text{(b)} \quad 2U\frac{\xi\eta}{r^4} - C\frac{\xi}{r^4 - 1} = 0. \end{aligned} \tag{5}$$

Since ξ appears as a factor on the left side of equation (5b), $\xi = 0$ or the η- axis is a solution to our problem; that is, a pair of vortices whose center points possess the coordinates η and $-\eta$ on the η- axis can remain stationary with respect to the cylinder. The force of the vortex belonging to the arbitrarily selected η can be calculated from (5). However, this solution for equations (5) has no further interest for us since they were not observed during the experiments. We seek further solutions for /4

the equations (5) or the determinant equation resulting from (5):

$$\begin{vmatrix} 1-\dfrac{\xi^2-\eta^2}{r^4} & \eta\left(\dfrac{1}{(r^2-1)^2}-\dfrac{1}{2\,r^2}\right) \\ \dfrac{2\,\eta}{r^4} & -\dfrac{1}{r^4-1} \end{vmatrix} = 0. \tag{6}$$

Since the equation (6) is not changed when the signs of ξ and η are changed, the curve must run symmetrically in all four quadrants. Equation (6) can be transformed into the following simple form:

$$\pm 2\,\eta = r - \frac{1}{r}, \tag{7}$$

where the different signs before η belong to the two branches of the curve.[3]

Equation (7) permits a simple construction of the curve which is depicted in Figure 1 for positive ξ. It is emphasized that at the point $\xi=1$, the slope of the curve is less than 45° and at infinity possesses an aysymptote that is angled against the ξ- axis at less than 30°. The course of the curve for negative ξ is naturally the drawn curve mirrored along the η- axis. However, in the infinite, this branch of the curve has no meaning for the direction of the velocity U we have selected and depicted /5
in Figure 1 for a certain position of the vortex pair.

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3. The correctness of the transformation of equation (6) to equation (7) can be very simply proven by reversing equation (7), which can also be written $4\eta^2 r^2 = (r^2-1)^2$, and inserting it in equation (6). The left side of equation (6) is thus transformed in the following manner:

$$\begin{vmatrix} 1-\dfrac{1}{r^2}+\dfrac{2\,\eta^2}{r^4} & \eta\,\dfrac{1-2\,r^2}{(r^2-1)^2} \\ \dfrac{2\,\eta}{r^4} & -\dfrac{1}{r^4-1} \end{vmatrix} = -\frac{1-\dfrac{1}{r^2}+\dfrac{(r^2-1)^2}{2\,r^6}}{r^4-1} - \frac{(r^2-1)^2(1-2\,r^2)}{2\,r^6(r^2-1)}$$

$$= \frac{(-2\,r^6+2\,r^4-r^4+2\,r^2-1)-(r^4-2\,r^6-1+2\,r^2)}{2\,r^6(r^4-1)} = 0.$$

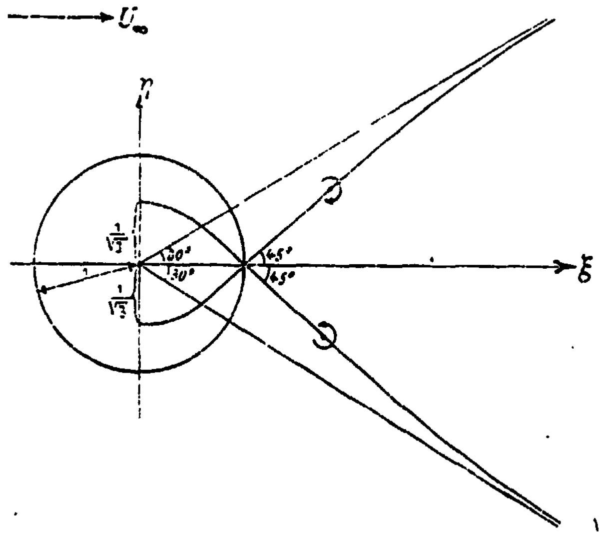

Figure 1

The force of the vortex C, determined by the various positions of the vortex pair along the curve is calculated from (5):

$$C = U \cdot 2\eta\left(1 - \frac{1}{r^4}\right). \qquad (8)$$

Accordingly, at a constant velocity U in the infinite, the vortex force C increases as the vortex pair lies at increasing distances along the curve away from the cylinder.

If we compare the accompanying photographs, then we find by measurement that the centers of the vortex pairs lie precisely along the curves determined above. From the hydrodynamic events observed when a very long circular cylinder is moved from a stationary position, we can make the following /6 sketch: A pair of vortices is detached from the rear edge of the cylinder, which, with continuously increasing growth of vortex force, approaches our curve, and after it has reached the curve, continues moving slowly along or next to the curve, while the vortex force increases as required by equation (8). Now the question is, how does the process develop further. The experiment shows that the development of the vortex pair as we have just followed it does not continue in the same fashion, but that the development of the flow changes completely. The two vortices abandon their symetrical positions with respect to the ξ- axis. New vortices are formed, and now alternately on either side of the cylinder; a kind of pendulous movement is set up in the region behind the cylinder, while the vortices detached earlier continue moving at some distance from the cylinder in two vortex paths. We now have a flow regime as investigated by Herr v. Karman [1]. The change of the original flow regime is evidently caused by an instability in the current. We shall therefore assign ourselves the task of testing the stability of the vortex pair behind the circular cylinder.

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2. Investigation of Stability of the Vortex Pair

We assume that the flow regime has developed to the extent that the center points of the two vortices lie on the curve determined in Section 1; we consider, that is, the condition illustrated in the photographs. As was already mentioned in Section 1, the vortex pair moves very slowly along the curve. Thus, for purposes of investigation of stability, we can view this condition as approximating the equilibrium condition. From this equilibrium condition, we will displace the vortex pair infinitesimally small distances /7 to see if the vortex pair returns to its equilibrium position. We shall conduct the investigation of stability in two parts: First, we will investigate stability when displacement from the equilibrium position has a mirror image symmetry; then, we will investigate asymmetric displacement.

To begin with the first case, we shall imagine the vortices displaced from their equilibrium position infinitesimally small amounts $+\alpha$ parallel to the ξ- axis and $+\beta$ or $-\beta$ parallel to the η- axis. Since the two vortices thus remain parallel to the ξ- axis, we can use the equations (4a) and (4b) derived for this condition, for the velocities $u = \frac{d\alpha}{dt}$ parallel to the ξ- axis and $v = \frac{d\beta}{dt}$ parallel to the η- axis. If in equations (4) one substitutes ξ with $\xi + \alpha$ and η with $\eta + \beta$ and develops the left side of equations (4) with the small quantities α and β, retaining only the first potentials of α and β, one obtains, taking the equations (5a) and (5b) into consideration (which cause the finite terms which appear during development to disappear by themselves) we have the following stability equations:

$$\text{(a)} \qquad A\alpha + B\beta = \frac{d\alpha}{dt},$$

$$\text{(b)} \qquad X\alpha + Y\beta = -\frac{d\beta}{dt}, \tag{9}$$

where

$$A = -\frac{2U}{r^6}\xi(4\eta^2 + 1); \quad B = \frac{2U}{r^5}\left(r^4 + 2r^2 + \frac{2\eta^2}{r^2}\right);$$
$$X = 8\xi^2\eta\frac{U}{r^6(r^4 - 1)}; \quad Y = \frac{2\xi U}{r^4}\left(1 + \frac{4\eta^2}{r^2(r^4 - 1)}\right). \tag{10}$$

If one eliminates one of the two dependent α or β from the two first order linear simultaneous differential equations (9a) and (9b), one obtains the following second order differential equation which is valid for β as well as for α: /8

$$\frac{d^2\alpha}{dt^2}+(Y-A)\frac{d\alpha}{dt}+(BX-AY)\alpha=0, \tag{11}$$

whose general solution is

$$\alpha=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t} \tag{12}$$

where

$$\lambda_{1,2}=-\frac{Y-A}{2}\pm\frac{1}{2}\sqrt{(Y-A)^2-4(BX-AY)}. \tag{13}$$

The conditions for stability are thus the two inequalities:

$$\begin{aligned}&\text{(a)} & Y&>A\\ &\text{(b)} & BX-AY&>0,\end{aligned} \tag{14}$$

which are met, as one can prove by substituting A, B, X, Y with values from (10).

The stability of the vortex pair for mirror image symmetrical displacement from the equilibrium position is thus proven. If one imagines a wall in place of the ξ- axis, so that fluid is found only on one side, then the single vortex is completely stable with respect to infinitesimal displacement from the equilibrium position, since the other three vortices are created as mirror images on the wall or on the cylinder as the case may be, and the arbitrary displacement must remain mirror-image symmetrical. Experiments with separating walls behind the cylinder have not as yet been dealt with.

Investigation of stability of the vortex pair with regard to assymetrical displacement from the equilibrium position remains to be discussed. If the coordinates of the equilibrium position of the first vortex are $\xi' \eta'$ and for

the second vortex $\xi' - \eta$, then the corresponding coordinates for the displaced vortex pair are $\xi + \alpha,\ \eta + \beta$ or $\xi - \alpha,\ -\eta + \beta$ where α and β are again /9 taken to mean infinitesimally small distances. In this case, we cannot use the equations (4) as above, since, like equations (1) and (3), they are valid only for mirror image symmetrical positions of the vortices. However, if in general, we characterize the coordinates of the vortex centers with $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, then by similar considerations that yield equations (1) and (3), we easily find for the complex displacement velocity of the first vortex:

$$u - iv = U\left(1 - \frac{1}{\zeta^2}\right) + iC\left\{\frac{1}{\zeta_1 - \frac{1}{\bar{\zeta}_2}} - \frac{1}{\zeta_1 - \zeta_2} - \frac{1}{\zeta_1 - \frac{1}{\bar{\zeta}_1}}\right\}, \tag{15}$$

where again $\bar{\zeta}$ represents the conjugated imaginary value of ζ.

If we separate equation (15) into real and imaginary parts, substitute ξ_1 with $\xi + \alpha$, η_1 with $\eta + \beta$, ξ_2 with $\xi - \alpha$ and η_2 with $-\eta + \beta$ and solve again for α and β, then we obtain as above the two stability equations:

$$\begin{aligned} &\text{(a)} \qquad A'\alpha + B'\beta = \frac{d\alpha}{dt}, \\ &\text{(b)} \qquad X'\alpha + Y'\beta = -\frac{d\beta}{dt}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} A' = Y' &= \frac{2U}{r^6}\xi\left\{2(\xi^2 - \eta^2) - \frac{r^2+1}{2}\right\}, \\ B' &= \frac{2U}{r^6}\eta\left\{2(\xi^2 - \eta^2) + \frac{r^2-1}{2}\right\}, \\ X' &= \frac{2U}{r^6}\eta\left\{2(\eta^2 - \xi^2) - \frac{r^2-1}{2} - 2r^2\eta(r^2+1)\right\}. \end{aligned} \tag{17}$$

Just as we derived the equation (11) from the two differential equations (9), we can use the two differential equations (16) to derive the following simplified second order differential equation as a consequence of $A' = Y'$: /10

$$\frac{d^2\alpha}{dt^2} + (B'X' - A'^2)\alpha = 0, \tag{18}$$

whose general solution is

$$\alpha = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

with

$$A'^2 - B'X' < 0 \tag{19}$$

so that

$$\lambda_{1,2} = \pm \sqrt{A'^2 - B'X'}, \tag{20}$$

as the condition of stability. By inserting the values of A', B', X' from (17), one can easily prove that the inequality (20) is not met for any point on the curve, so that the unstability of our vortex pair is proved. The two root values

$$\lambda_1 = \sqrt{A'^2 - B'X'} \quad \text{and} \quad \lambda_2 = -\sqrt{A'^2 - B'X'}$$

belong to the two principal oscillations of the vortex pair for asymmetrical displacements. The value corresponding to λ_1 gives rise to instability, while the principal oscillation corresponding to λ_2 represents a dampened movement. From the first principal oscillation, one obtains from (16) the relationship

$$\text{(a)} \qquad \frac{\beta}{\alpha} = -\frac{A' - \sqrt{A'^2 - B'X'}}{B'};$$

and for the latter, the stable motion, (21)

$$\text{(b)} \qquad \frac{\beta}{\alpha} = -\frac{A' + \sqrt{A'^2 - B'X'}}{B'}.$$

However, since according to (17) $B'X'$ is always negative, B' always positive, and X' always negative, one sees that in the case of unstable displacement $\frac{\beta}{\alpha}$ is positive, and in the stable case $\frac{\beta}{\alpha}$ is negative, and taken absolutely, is greater than the first case. If one uses the above formulas (21) for $\frac{\beta}{\alpha}$ of the specific position of the vortex pair in the photographs, one obtains after simple calculation: /11

$$\frac{\beta}{\alpha} = 1.2 \quad \text{for unstable displacement}$$

$$\frac{\beta}{\alpha} = -3.7 \quad \text{for stable displacement.}$$

The change of the observed current formation which was discussed at the end of Section 1 thus has its more fundamental source in the instability of one

of the four principal oscillations of the vortex pair. Indeed, one observes that soon after the vortex pair has reached our curve and starts to move very slowly, there is a noticeable change in the hydrodynamic process which is caused by the instability just calculated. However, this instability does not appear as a sudden asymmetric displacement of the potential of the two vortices as one might expect from the preceding calculation; rather, before the unstable displacement commences, both vortices lose their characteristics as potential vortices, where a spinning nucleus forms in the center of the vortex which is deformed into an elipse and occasionally also divides. Only after this deformation of the vortex does the displacement referred to above occur, which starts the formation of the two parallel vortex streams as described by Kármán.

3. The Resistance of the Cylinder

We shall propose the question: Is the resistance of the moving cylinder theoretically explained by the observed movement of the vortex pair away from the cylinder? For this purpose, we shall imagine the cylinder moving with uniform motion and shall calculate the chronological change of the total impulse caused by the displacement of the vortex pair with respect to the cylinder. If the vortex pair remained stationary with respect to the uniformly moving cylinder, then the resistance of the cylinder would be zero. Indeed, however, one observes that the vortices grow and concurrently move /12
on. We shall mathematically grasp this process by assuming that there is a potential flow outside the vortex nuclei, but that we nevertheless think of the vortex as chronologically growing. The vortex pair is at this time moving along the curve determined in Section 1. The total impulse is divided into two parts: the impulse of the potential flow around the circular cylinder and that of the vortex pair. The first part is $U \cdot \pi \cdot \varrho$, when ϱ is the density of the fluid, U the velocity of the cylinder, and the circular radius is 1 [2]. The second part stems from the two solids and the two mirrored vortices. The complex potential of these four vortices with the coordinates $\zeta_1\ \zeta_2\ \zeta_3\ \zeta_4$ (see Figure 2) is:

$$W = \Phi + i\Psi = Ci \lg \frac{(\zeta - \zeta_1)(\zeta - \zeta_4)}{(\zeta - \zeta_2)(\zeta - \zeta_3)}, \qquad (22)$$

so that

$$\text{(a)} \qquad \Phi = C(\alpha - \beta)$$
$$\text{(b)} \qquad \Psi = C \lg \frac{r_1 r_4}{r_2 r_3}. \tag{23}$$

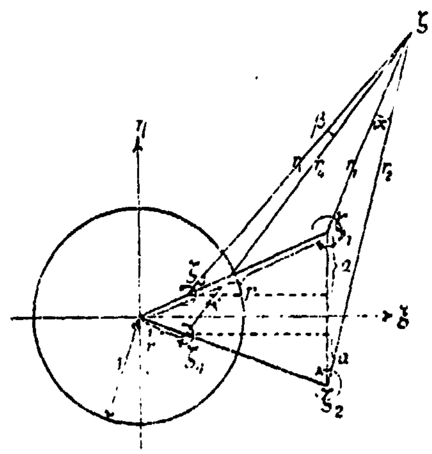

Figure 2

The X components of the impulse of the total fluid are /13

$$P = \varrho \iint u \, dx \, dy = \varrho \iint \frac{\partial \Phi}{\partial x} \, dx \, dy. \tag{24}$$

where the integration extends over that part of the plane which is filled with fluid.[4] The integration in terms of can be carried out in the last double integral, and taking (23a) into consideration, one obtains

$$P = 4\pi C \varrho - C \varrho \int_{(\text{circle})} (\alpha - \beta) \, dy. \tag{25}$$

α here has the meaning indicated in Figure 2, and the integral extends over the circumference of the unit circle. The latter integral, which can also be written by substitution of $y = \sin \gamma$, thus:

$$J = \int_0^{2\pi} (\alpha - \beta) \cos \gamma \, d\gamma$$

can most simply be solved by means of complex integration. We separate J into its two parts which stem from its two pairs of vortices:

$$J_1 = \int_0^{2\pi} \alpha \cos \gamma \, d\gamma, \qquad J_2 = \int_0^{2\pi} \beta \cos \gamma \, d\gamma,$$

so that

$$J = J_1 - J_2,$$

and to determine J_1, we calculate the complex integral

4. Correspondingly, one obtains for the Y components the impulse $Q = \varrho \iint v \, dx \, dy$. However, one can immediately see from reasons of symmetry that $Q = 0$.

$$J_1' = \int_0^{2\pi} \lg \frac{e^{i\gamma} - re^{i\varphi}}{e^{i\gamma} - re^{-i\varphi}} e^{i\gamma} d\gamma; \quad J_1'' = \int_0^{2\pi} \lg \frac{e^{i\gamma} - re^{i\varphi}}{e^{i\gamma} - re^{-i\varphi}} e^{-i\gamma} d\gamma,$$

so that

$$J_1' + J_1'' = 2\int_0^{2\pi} \lg \frac{e^{i\gamma} - re^{i\varphi}}{e^{i\gamma} - re^{-i\varphi}} \cos\gamma \, d\gamma.$$

The meaning of the quantities φ and r can be seen in Figure 2. The /14 imaginary part $J_1' + J_1''$ is doubled. The two integrals J_1' and J_1'' can, however, be easily calculated if one introduces $\zeta = e^{i\gamma}$ as an integration variable into the first integral and $\zeta = e^{-i\gamma}$ into the second; then:

$$J_1' = -i\int_{\text{(circle)}} \lg \frac{\zeta - re^{i\varphi}}{\zeta - re^{-i\varphi}} d\zeta; \qquad J_1'' = -i\int_{\text{(circle)}} \lg \frac{1 - \zeta re^{i\varphi}}{1 - \zeta re^{-i\varphi}} d\zeta.$$

Since and $re^{-i\varphi}$ lie outside the path of the integration, then $J_1' = 0$. In contrast, J_1'' is different from zero in that

$$\int_{\text{(circle)}} \lg (1 - \zeta re^{i\varphi}) d\zeta = -\frac{2i\pi}{re^{i\varphi}},$$

$$\int_{\text{(circle)}} \lg (1 - \zeta re^{-i\varphi}) d\zeta = -\frac{2i\pi}{re^{-i\varphi}},$$

so that

$$J_1'' = -\frac{2\pi}{r}(e^{-i\varphi} - e^{i\varphi}) = \frac{4\pi i}{r} \sin\varphi.$$

Thus is

$$J_1 = \frac{2\pi}{r} \sin\varphi.$$

The same method is used to calculate J_2, and the same formulas are valid if one simply substitutes r with $\frac{1}{r}$. Correspondingly, one obtains

$$J_2 = 2\pi r \sin\varphi,$$

so that

$$J = J_1 - J_2 = -2\pi \sin\varphi \left(r - \frac{1}{r}\right)$$

or

$$J = -2\pi a$$

since

$$\sin\varphi\left(r - \frac{1}{r}\right) = a,$$

as follows from Figure 2. Equation (25), which gives the part stemming /15

from the vortices is now

$$P = 4 C a \pi \varrho + 2 C a \pi \varrho = 6 C a \pi \varrho. \tag{26}$$

If one now includes the impulse of the potential flow, then one obtains the total impulse:

$$G = U \pi \varrho + 6 C a \pi \varrho. \tag{27}$$

The fluid pressure on the cylinder is determined by a momentary change of C ; and at constant velocity is

$$\frac{dG}{dt} = 6 \pi \varrho \frac{d(Ca)}{dt}. \tag{28}$$

Since according to equation (8), can be expressed by the momentary position of the vortex on our curve, then according to (28), the pressure experienced by the cylinder at constant velocity can be derived from the position and the momentary change of position of the vortex pair.

Great difficulties stand in the way of calculating the resistance on the basis of the observed movement of the vortex pair along our curve. This is largely because of the instabilityof the vortex proven in Section 2, where soon after the vortex pair has reached the curve, it leaves it again as a consequence of an unstabledisturbance. If, however, two vortex streams have formed behind the cylinder as per the vortex formation of v. Kármán. then, as v. Kármán /3/ has shown, the resistance of the body can be determined with satisfying accuracy from observation of the displacement velocity of the vortex, as well as the distance between the vortex streams.

4. The Flow Around the Infinitely Long, Flat Plate /16

The observations made in Sections 1 through 3 for a circular cylinder suggest expanding the concepts to the motion of a flat plate. We will assume that the direction of movement is perpendicular to the plate. It is very easy to prove that behind a moving plate there must also be a geometric location for the position of the two vortices turning against the plate with opposite rotation. The reason is that if we form the previously considered

flow around the unit circle of the ζ plane together with the vortex pair through

$$z = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right) \tag{29}$$

conforming to z plane of the plate of length 2 and form the derivation of the complex velocity potential for the flow in the z plane, which is obtained from equation (1) by means of elimination of ζ using equation (29), then the condition for the vortex pair to remain stationary behind the plate is

$$\frac{dW}{dz} = \frac{dW}{d\zeta} \cdot \frac{d\zeta}{dz} = o. \tag{30}$$

However, since everywhere in the layer of the z plane that the physical condition $\frac{d\zeta}{dz} \neq o$ occurs, equation (30) simplifies to

$$\frac{dW}{d\zeta} = o.$$

However, this condition is the same as those expressed in equations (5) of Section 1. That is, we obtain the desired geometric location when we express the curve determined through equation (7) in the coordinates $z = x + iy$ with the help of equation (29).

I will not carry out this transformation; rather, I shall deal with the most interesting matter of whether there exist positions of the vortex pair behind the plate that effect finite velocities at either end of the plate. /17
The calculation shows that this condition is met only for the position of the vortex pair in the infinite, which is naturally out of the question for the explanation of the flow formation. Observations of the flow accompanying a moving plate show that similar to a cylinder, soon after movement commences, a vortex pair is detached from the rear edge of the plate, which then steadily grows and moves away from the plate.

If one uses instead of a plate, a bowl with a cross section of a circular arc and moves it with the convex side forward, then my theory, as

Professor Prandtl pointed out to me, predicts finite velocities at either end of the bowl, for a certain position of the vortex pair in the finite and depending on the flexure of the bowl. I will not elaborate any further on this question as no photographs of this flow configuration are at hand.

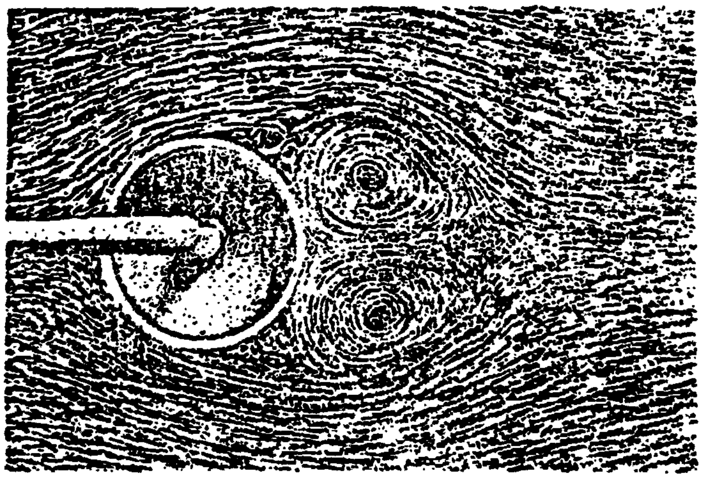

Figure 3

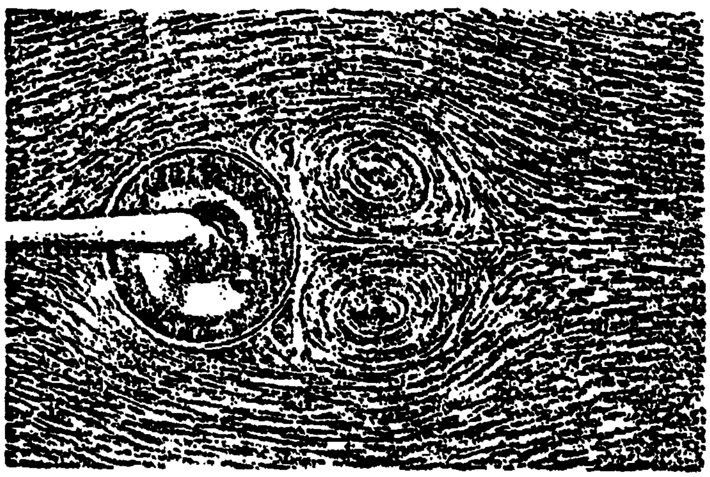

Figure 4

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2. Lamb, S., Lehrbuch der Hydronomik, translated by v. Friedel, p. 93.

3. v. Kármán and Rubach, Phys. Zeitschr. 1912.